KANYASHREE UNIVERSITY

M.Sc. 2nd Semester Examination-2024

Subject: Mathematics

Course- CC 8

Functional Analysis

Full Marks-40 **Time-2.00 Hours**

GROUP - A

(Answer **any four** of the following) **(5×4=20)**

1. Define equivalent norm in a normed linear space. Prove that any two norms in a finite dimensional normed linear space are equivalent. 1+4
2. Define isometric spaces. Show that every real linear space X with dimension n is isomorphic to the Euclidean n- space $\mathbb{R}^n$. 1+4
3. Given that x_0 be a nonzero vector in a normed linear space X. Show that there is an $f \in X^*$ such that $\|f\| = 1$ and $(x_0) = \|x_0\|$. 5
4. Prove that for each bounded linear functional f on a Hilbert space H, there is a unique vector $z \in H$ such that $f(x) = (x, z)\ \forall x \in H$ and further $\|f\| = \|z\|$. 5
5. Proved that a normed linear space X is a Banach space if and only if the set $S = \{x \in X: \|x\| = 1\}$ is complete. 5
6. Let (X, d) be a non-empty complete metric space. Show that every contraction mapping $T: (X, d) \to (X, d)$ has a unique fixed point. 5

GROUP – B

(Answer **any two** of the following) **(10× 2=20)**

1. i) State and prove Riesz lemma.

   ii) Show that the inner product space is a normed linear space under the norm $\|.\|$ defined by $\|x\| = \sqrt{(x, x)}\ \forall x \in X$.

   iii) Let $T: X \to Y$ be a linear operator. Show that T^{-1} is a linear operator on $T(x)$, provided T^{-1} exists. 5+3+2

2. i) State and prove open mapping theorem.

   ii) Prove that in any normed linear space X, all open balls are convex.

   iii) Give an example of a linear metric space which is not a normed linear space. 5+3+2

3. i) State and prove Bessel's inequality.

   ii) Let $T: X \to Y$ be a bounded linear operator, where X and Y are normed linear spaces over the same scalar field. Show that $\|T\| = \sup\{\|Tx\|: \|x\| \leq 1\}$.

   iii) Is the space $C[a, b]$ a Hilbert space? Justify your answer. 4+4+2

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